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# Positive Operator Method to Establish Principle of Exchange of Stabilities in Thermal Convection of a Viscoelastic Fluid

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**Abstract**— The theoretical treatments of convective stability problems usually invoked the so-called principle of exchange of stabilities (PES), which is demonstrated physically as convection occurring initially as a stationary convection. Weinberger [1969] used a method of a positive operator, a generalization of a positive matrix operator, to establish the PES, wherein, the resolvent of the linearized stability operator is analyzed, which is a composition of certain integral operators. Motivated by method of positive operator of Weinberger, we aim to extend this analysis of Herron[2000] to establish the PES to more general convective problems from the domain of non-Newtonian fluid. In the present paper, the problem of heated from below with variable gravity is analyzed by the method of positive and it is established that PES is valid for this general problem, when the gravity is a nonnegative throughout the fluid layer and the elastic number of the medium is less than the ratio of permeability to porosity.

**Index Terms**— Principle of exchange of stabilities; Stationary convection; Positive Operator; Stability Operator; Resolvent; Integral Operator; Viscoelastic fluid.

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## 1 INTRODUCTION

Rayleigh–Bénard convection is a fundamental phenomenon found in many atmospheric and industrial applications. The problem has been studied extensively experimentally and theoretically because of its frequent occurrence in various fields of science and engineering. This importance leads the authors to explore different methods to study the flow of these fluids. Many analytical and numerical methods have been applied to analyze this problem in the domain of Newtonian fluids, including the linearized perturbation method, the lattice Boltzmann method (LBM), which has emerged as one of the most powerful computational fluid dynamics (CFD) methods in recent years.

A problem in fluid mechanics involving the onset of convection has been of great interest for some time. The theoretical treatments of convective problems usually invoked the so-called principle of exchange of stabilities (PES), which is demonstrated physically as convection occurring initially as a stationary convection. This has been stated as "all non decaying disturbances are non oscillatory in time". Alternatively, it can be stated as "the first unstable eigenvalues of the linearized system has imaginary part equal to zero". Mathematically, if $\sigma_r \geq 0 \Rightarrow \sigma_i = 0$ (or equivalently, $\sigma_i \neq 0 \Rightarrow \sigma_r < 0$), then for neutral stability $(\sigma_r = 0)$, $\sigma = 0$, where $\sigma_r$ and $\sigma_i$ are respectively the real and imaginary parts of the complex growth rate $\sigma$ . This is called the 'principle of exchange of stabilities' (PES). The establishment of this principle results in the elimination of unsteady terms in a certain class of stability problems from the governing linerized perturbation equations. Further, we know that PES also plays an important role in the bifurcation theory of instability.

Pellew and Southwell [1] took the first decisive step in the direction of the establishment of PES in Rayleigh-Bénard convection problems in a comprehensive manner. S. H. Davis [2] proved an important theorem concerning this problem. He proved that the eigenvalues of the linearized stability equations will continue to be real when considered as a suitably small perturbation of a self-adjoint problem, such as was considered by Pellew and Southwell. This was one of the first instances in which *Operator Theory* was employed in hydrodynamic stability theory. As one of several applications of this theorem, he studied Rayleigh-Bénard convection with a constant gravity and established PES for the problem. Since then several authors have studied this problem under the varying assumptions of hydromagnetics and hydrodynamics.

Convection in porous medium has been studied with great interest for more than a century and has found many applications in underground coal gasification, solar energy conversion, oil reservoir simulation, ground water contaminant transport, geothermal energy extraction and in many other areas. Also, the importance of non-Newtonian fluids in modern technology and industries is ever increasing and currently the stability investigations of such fluids are a subject matter of intense research. A non–Newtonian fluid is a fluid in which viscosity changes with the applied strain rate and as a result of which the non-Newtonian fluid may not have a well-defined viscosity. Viscoelastic fluids are such fluids whose behaviour at sufficiently small variable shear stresses can be characterized by three constants viz. a co-efficient of viscosity, a relaxation time and a retardation time, and

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whose invariant differential equations of state for general motion are linear in stresses and include terms of no higher degree than the second in the stresses and velocity gradients together. The problem of the onset of thermal instability in a horizontal layer of viscous fluid heated from below has its origin in the experimental observation of Bénard [3]. Oldroyd [4] proposed and studied the constitutive relations for viscoelastic fluids in an attempt to explain the rheological behavior of some non-Newtonian fluids. Since then numerous research papers pertaining to the stability investigations of non-Newtonian fluids under the effects of different external force fields and in presence of porous medium have been reported. Shenoy [5] had reviewed studies of flow in non-Newtonian fluids in porous medium, with attention concentrated on power-law fluids. For further reviews of the fundamental ideas, methods and results concerning the convective problems from the domain of Newtonian/ non-Newtonian fluids, one may be referred to Chandrasekhar [6], Lin [7], Drazin and Reid [8] and Nield and Bejan [9].

It is clear from the above discussion that the Pellew and Southwell method is a useful and simple tool for the establishment of PES in convective problems when the resulting eigenvalue problem, in terms of differential equations and boundary conditions, is having constant coefficients. Thus, the method is not always useful to determine the PES for those convective problems, which are either permeated with some external force fields, such as variable gravity, magnetic field, rotation etc., are imposed on the basic Thermal Convection problems and the resulting eigenvalue problems contain variable coefficient/s or an implicit function of growth rate as in the case of non-Newtonian fluids. The present work is partly inspired by the above discussions, and the works of Herron [10], [11] and the striking features of convection in non-Newtonian fluids in porous medium and motivated by the desire to study the above discussed problems.

Our objective here is to extend the analysis of Weinberger [12] and Rabinowitz [13] based on the method of positive operator to establish the PES to these more general convective problems from the domain of non-Newtonian fluid. In the present paper, the problem of Thermal convection of a viscoelastic fluid in porous medium heated from below with variable gravity is analyzed; and using the positive operator method, it is established that PES is valid for this problem, when g (z) (the gravity field) is nonnegative throughout the fluid layer and the elastic number of the medium is less than the ratio of permeability to porosity, i.e. $\Gamma < \frac{P_l}{\varepsilon}$ or $\lambda < \frac{k_1}{\varepsilon\upsilon}$.

## 2 Mathematical Formulation of the Physical Problem

Consider an infinite horizontal porous layer of viscoelastic fluid of depth 'd' confined between two horizontal planes z = 0 and z = d under the effect of variable gravity, $\vec{g}(0,0-g(z))$. Let $\Delta T$ be the temperature difference between the lower and upper plates. The fluid is assumed to be viscoelastic and described by the Oldroydian constitutive equations. Thus, the governing equations for the Rayleigh-Bénard situation in a viscoelastic fluid–saturated porous medium under Boussinesq approximation and under the effect of variable gravity are (see [6] & [9]);

$$\frac{1}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\vec{q}}{\partial t}$$

$$=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\frac{\nabla p}{\rho_0}+\left(1+\frac{\delta\rho}{\rho_0}\right)\vec{X}\right]-\left(1+\lambda_0\frac{\partial}{\partial t}\right)\left(\frac{\upsilon}{k_1}\right) \quad (1)$$

$$\nabla.\vec{q}=0 \quad (2)$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T = K\nabla^2 T \quad (3)$$

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right] \quad (4)$$

In the above equations, $\vec{q}$, T $\rho$, K, $\alpha$ $\lambda, \lambda_0$ and $\upsilon$ stand for filter velocity, temperature, density, thermal diffusivity, coefficient of thermal expansion, the relaxation time and the retardation time and the kinematic viscosity, respectively. Here, $E=\varepsilon+(1-\varepsilon)\frac{\rho_s c_s}{\rho_0 c_v}$ is a constant, where $\rho_{s,} c_s$ stand for density and heat capacity of solid (porous matrix) material and $\rho_{0,}, c_v$ for fluid, respectively. Here, the suffix zero refers to the value at the reference level z = 0. This is to mention here that, when the fluid slowly percolates through the pores of the rock, the gross effect is represented by the usual Darcy`s law. As a consequence, the usual viscous terms has been replaced by the resistance term $\left(-\frac{\mu}{k_1}\right)\vec{q}$ in the above equations of motion. Here, $\mu$ and $k_1$ are the viscosity and the permeability of the medium.

Following the usual steps of the linearized stability theory, it is easily seen that the nondimensional linearized perturbation equations governing the physical problem described by equations (1)-(4) can be put into the following forms, upon ascribing the dependence of the perturbations of the form $\exp\left[i\left(k_x x+k_y y\right)+\sigma t\right]$, $(\sigma=\sigma_r+i\sigma_i)$ (c.f. Chandrasekhar [6] and Siddheshwar and Krishna [14]);

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right)\left(D^2-k^2\right)w=-g(z)Rk^2\theta \quad (5)$$

$$\left(D^2-k^2-\sigma E P_r\right)\theta=-w \quad (6)$$

together with following dynamically free, thermally and electrically perfectly conducting boundary conditions;

$$w=0=\theta=D^2 w \quad \text{at } z=0 \text{ and } z=1 \quad (7)$$

In the forgoing equations; z is the real independent variable, $D\equiv d/dz$ is the differentiation with respect to $z$, $k^2$ is the square of the wave number, $Pr=\frac{\upsilon}{\kappa}$ is the thermal Prandtl number, $P_l=\frac{k_1}{d^2}$ is the dimensionless medium permeability, $\Gamma=\frac{\lambda\upsilon}{d^2}$ is elastic number, E= $\varepsilon+(1-\varepsilon)\frac{\rho_s c_s}{\rho_0 c_v}$ is constant, $R^2=\frac{g_0\alpha\beta d^4}{\kappa\upsilon}$ is the thermal Rayleigh number, $\sigma=\sigma_r+i\sigma_i$ is the complex growth rate associated with the perturbations and $w,\theta$ are the perturbations in the vertical velocity, temperature, respectively.

The system of equations (5)-(6) together with the boundary conditions (7) constitutes an eigenvalue problem for σ for the given values of the parameters of the fluid and a given state of the system is stable, neutral or unstable according to whether $\sigma_r$ is negative, zero or positive.

It is remarkable to note here that equations (5)-(6) contain a variable coefficient and an implicit function of $\sigma$, hence as discussed earlier the usual method of Pellew and Southwell is not useful here to establish PES for this general problem. Thus, we shall use the method of positive operator to establish PES.

## 3 ABSTRACT FORMULATION

### The Method of Positive Operator

We seek conditions under which solutions of equations (5)-(6) together with the boundary conditions (7) grow. The idea of the method of the solution is based on the notion of a 'positive operator', a generalization of a positive matrix, that is, one with all its entries positive. Such matrices have the property that they possess a single greatest positive eigenvalue, identical to the spectral radius. The natural generalization of a matrix operator is an integral operator with non-negative kernel. To apply the method, the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of certain integral operators. When the Green's function Kernels for these operators are all nonnegative, the resulting operator is termed positive. The abstract theory is based on the Krein – Rutman theorem [15], which states that;

"If a linear, compact operator A, leaving invariant a cone $\hbar$, has a point of the spectrum different from zero, then it has a positive eigenvalue $\lambda$, not less in modulus than every other eigenvalue, and this number corresponds at least one eigenvector $\varphi\in\hbar$ of the operator A, and at least one eigenvector $\phi\in\hbar^*$ of the operator $A^*$". For the present problem the cone consists of the set of nonnegative functions.

To apply the method of positive operator, formulate equations (5) and (6) together with boundary conditions (7) in terms of certain operators as;

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{p_l}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right]Mw=g(z)R\theta k^2 \quad (8)$$

$$\left(M+\sigma E\,Pr\right)\theta=Rw \quad (9)$$

where,

$Mw=mw, w\in domM; M^2w=m^2w, w\in dom\left(MM\right)$;

and $M\theta=m\theta,\ w\in domM$

The domains are contained in B, where

$$B=L^2\left(0,1\right)=\left\{\varphi\ \middle|\ \int_0^1|\varphi|^2dz<\infty\right\},$$

with scalar product $\langle\varphi,\phi\rangle=\int_0^1\varphi\left(z\right)\overline{\phi\left(z\right)}dz$, $\varphi,\phi\in B$; and norm $\|\varphi\|=\langle\varphi,\varphi\rangle^{1/2}$.

We know that $L^2(0,1)$ is a Hilbert space, so, the domain of M is

$dom\,M=\left\{\varphi\in B/\ D\varphi, m\varphi\in B,\ \varphi\left(0\right)=\varphi\left(1\right)=0\right\}$.

We now formulate the homogeneous problem corresponding to equations (5)-(6) by eliminating $\theta$ from (8) and (9) as;

$$w=k^2R^2M^{-1}\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right)^{-1}g\left(z\right)\left(M+E\,Pr\,\sigma\right)^{-1}w \quad (10)$$

or $w=K\left(\sigma\right)w$, (11)

where,

$$K(\sigma)=k^2R^2T(0)\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right)^{-1}g\left(z\right)T\left(E\,Pr\,\sigma\right)w \quad (12)$$

In equation (12), we have defined

$T\left(E\,Pr\,\sigma\right)=\left(M+E\,Pr\,\sigma\right)^{-1}$

and this exists for

$$\sigma\in T_{\frac{k}{\sqrt{Pr\,E}}}=\left\{\sigma\in C\ \middle|\ Re\left(\sigma\right)>\frac{-k^2}{E\,Pr}, Im\left(\sigma\right)=0\right\}$$

and $\|T\left(\sigma E\,Pr\right)\|^{-1}>\left|\sigma+\frac{k^2}{E\,Pr}\right|$ for $Re\left(\sigma\right)>-\frac{k^2}{E\,Pr}$.

Here, 'Re' and 'Im' respectively stand for real and imaginary part of the quantity.

Now, since $T\left(E\,Pr\,\sigma\right)$ is an integral operator such that for $f\in B$,

$$T\left(\sigma E\,Pr\right)f=\int_0^1 g\left(z,\xi;\sigma E\,Pr\right)f\left(\xi\right)d\xi,$$

where, $g\left(z,\xi,E\,Pr\,\sigma\right)$ is the Green's function kernel for the operator $\left(M+\sigma E\,Pr\right)$, which can be readily computed following Herron [2000] as;

$$g\left(z,\xi,Pr\,\sigma\right)=\frac{\cosh\left[r\left(1-|z-\xi|\right)\right]-\cosh\left[r\left(-1+z+\xi\right)\right]}{2r\sinh r}$$

where, $r = \sqrt{k^2 + \sigma E \Pr}$.

In particular, taking $\sigma = 0$, we have $M^{-1} = T(0)$ is also an integral operator.

$K(\sigma)$ defined in (12), which is a linear combination of certain integral operators, is termed as linearized stability operator. $K(\sigma)$ depends analytically on $\sigma$ in a certain right half of the complex plane. It is clear from the form of $K(\sigma)$ that it also contains an implicit function of $\sigma$.

We shall examine the resolvent of the $K(\sigma)$ defined as $\left[I - K(\sigma)\right]^{-1}$;

$$\left[I - K(\sigma)\right]^{-1} = I - \left[I - K(\sigma_0)\right]^{-1}\left[K(\sigma) - K(\sigma_0)\right]^{-1}\left[I - K(\sigma_0)\right]^{-1} \quad (13)$$

If for all $\sigma_0$ greater than some a,

(1) $\left[I - K(\sigma_0)\right]^{-1}$ is positive,

(2) $K(\sigma)$ has a power series about $\sigma_0$ in $(\sigma_0 - \sigma)$ with positive coefficients; i.e., $\left(-\frac{d}{d\sigma}\right)^n K(\sigma_0)$ is positive for all n, then the right side of (13) has an expansion in $\sigma_0 - \sigma$ with positive coefficients. Hence, we may apply the methods of Rabinowitz [12] and Weinberger [13], to show that there exists a real eigenvalue $\sigma_1$ such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma : \mathrm{Re}(\sigma) \le \sigma_1\}$.

This is result is equivalent to PES, which was stated earlier as "the first unstable eigenvalue of the linearized system has imaginary part equal to zero."

## 4. THE PRINCIPLE OF EXCHANGE OF STABILITIES (PES)

As operator $T(E \Pr \sigma) = (M + E \Pr \sigma)^{-1}$ is an integral operator whose kernel $g(z, \xi, E \Pr \sigma)$ is the Laplace transform of the Green's function $G(z, \xi; t)$ for the initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial z^2} + k^2 + E \Pr \frac{\partial}{\partial t}\right) G = \delta(z - \xi, t), \quad (14)$$

where, $\delta(z - \xi, t)$ is Dirac–delta function in two-dimension,

with boundary conditions

$$G(0, \xi; t) = G(1, \xi; t) = G(z, \xi; 0) = 0, \quad (15)$$

Following Herron [10], by direct calculation of the inverse Laplace transform, we can have

$T(\sigma E \Pr) = (M + \sigma E \Pr)^{-1}$ is a positive operator for all real $\sigma_0 > -\frac{k^2}{E \Pr}$, and that $T(\sigma E \Pr)$ has a power series about $\sigma_0$ in $(\sigma_0 - \sigma)$ with positive coefficients.

In other words;

for all real $\sigma_0 > -\frac{k^2}{E \Pr}$, we see that

$$\left(-\frac{d}{d\sigma}\right)^n g(z, \xi, \sigma E \Pr) = \int_0^\infty t^n e^{-\sigma E \Pr t} G(z, \xi, t)\, dt \ge 0$$

i.e. positive.

In particular, from the above result, taking $\sigma = 0$, we deduce that $T(0) = M^{-1}$ is also a positive operator.

Since, $K(\sigma)$ is a linear operator, Condition (1) can be easily verified by following the analysis of Herron [10] for $K(\sigma)$, i.e. $K(\sigma)$ is a linear, compact integral operator, and has a power series about $\sigma_0$ in $(\sigma_0 - \sigma)$ with positive coefficients. Thus, $K(\sigma)$ is a positive operator leaving invariant a cone (set of non-negative functions).

Moreover, for $\sigma$ real and sufficiently large, the norms of the operators $T(0)$ and $T(E \Pr \sigma)$ become arbitrarily small. So,

$\|K(\sigma)\| < 1$.

Hence, $\left[I - K(\sigma)\right]^{-1}$ has a convergent Neumann series, which implies that $\left[I - K(\sigma)\right]^{-1}$ is a positive operator. This is the content of Condition (1).

To verify Condition (2), we see that

$\left(\frac{\sigma_0}{\varepsilon} + \frac{1}{p_1}\left(\frac{1 + \Gamma\mu\sigma_0}{1 + \Gamma\sigma_0}\right)\right)^{-1} > 0$ for all $\sigma_0$ real, and

$\sigma_0 > \sqrt{\frac{p_1 + \mu\varepsilon\Gamma}{2\Gamma p_1} - \frac{\varepsilon}{\Gamma p_1}} - \frac{p_1 + \mu\varepsilon\Gamma}{2\Gamma p_1}$ and $\Gamma < \frac{p_1}{\varepsilon}$ or $\lambda < \frac{k_1}{\varepsilon \upsilon}$.

Therefore, for all real

$$\sigma_0 > \max\left(-\frac{k^2}{E \Pr}, \sqrt{\frac{p_1 + \mu\varepsilon\Gamma}{2\Gamma p_1} - \frac{\varepsilon}{\Gamma p_1}} - \frac{p_1 + \mu\varepsilon\Gamma}{2\Gamma p_1}\right),$$

$g(z) \ge 0$ for all $z \in [0,1]$, by the product rule for differentiation one concludes that $K(\sigma)$ in (12) satisfies Condition (2).

Hence, by the Krein–Rutman theorem, we have the following result;

**Theorem**. *PES holds for (5) - (6) together with boundary conditions (7) when g (z) is nonnegative throughout the layer and* $\Gamma < \frac{P_1}{\varepsilon}$ or $\lambda < \frac{k_1}{\varepsilon \upsilon}$.

## Conclusions

Thus, we see from the present analysis that PES can be established for this general convective problem from the domain of the non-Newtonian fluid using the method of positive operator, and thus extend the analysis of Weinberger [13] to the domain of non-Newtonian fluids. Theorem proves that, when g(z) (the gravity field) is nonnegative throughout the fluid layer and the elastic constant of the medium is less than the ratio of permeability to porosity, i.e. $\Gamma < P_l/\varepsilon$ or equivalently $\lambda < k_1/\varepsilon\upsilon$, PES is valid.

In particular, letting $\Gamma = 0$, one recover the result of Herron[10] for Bénard convection problem with variable gravity field.

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